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Strong Converse Inequalities*

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Converse inequalities are proved for a family of operators that state the equivalence of two terms of error in approximation to the revelant modulus of smoothness. Such inequalities have been proved by Z. Ditzian and K. G. Ivanov with a different method. Our emphasis is that these so-called strong converse inequalities follow from some standard estimates on the derivatives of the operators without additional work; hence we extend the Ditzian–Ivanov result to a large family of operators. The method of the paper is very close in spirit to the classical parabola technique.

Let $f \in C[0, 1]$,

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

the associated Bernstein polynomials and with $\varphi(x) = \sqrt{x(1-x)}$

$$\omega_\varphi(f; \delta) = \sup_{0 \leqslant t \leqslant \delta} \|\Delta^2_{t\varphi} f\|$$
$$= \sup_{0 \leqslant t \leqslant \delta} \sup_x |f(x - t\varphi(x)) - 2f(x) + f(x + t\varphi(x))| \qquad (1)$$

the second modulus of smoothness with step-weight function φ, where the second supremum is taken for those values x for which every argument belongs to $[0, 1]$ (we mention that the standard notation for ω_φ would be ω^2_φ—see, e.g., [2]—but for the sake of simplicity we drop the upper index because we shall always work with moduli of smoothness of order 2). The estimate

$$\|B_n(f) - f\| \leqslant C\omega_\varphi\left(f; \frac{1}{\sqrt{n}}\right), \qquad n = 1, 2, \ldots$$

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with some absolute constant C is well known (see, e.g., [2]), but in the other direction only Stechkin-type estimates have been used in the literature (see [2]). In a recent breakthrough, Z. Ditzian and K. G. Ivanov [1] verified that actually there is a K such that

$$\omega_\varphi\left(f;\frac{1}{\sqrt{n}}\right)\leqslant C\frac{m}{n}\left(\|B_n(f)-f\|+\|B_m f-f\|\right),\qquad n=1,2,...$$

holds for every $m\geqslant Kn$. Thus, by choosing $m=Kn$ we get the equivalence

$$\omega_\varphi\left(f;\frac{1}{\sqrt{n}}\right)\sim\|B_n(f)-f\|+\|B_{Kn}f-f\|,$$

where $\sim$ means that the ratio of the two sides remains in between two positive absolute constants.

Ditzian and Ivanov combined a new estimate involving the Voronovskaya of $B_n(f)$ with some more familiar ones to get the above converse result. The aim of this paper is to show that the same goal can be achieved by using some well-known estimates, thereby we extend the above converse result to a family of more general operators.

In fact, let $\{L_n\}$ be a sequence of positive operators acting on the continuous functions defined on some interval I. Typical examples are the Bernstein operators mentioned above, the Szász–Mirakjan operators

$$S_n(f;x)=\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)e^{-nx}\frac{(nx)^k}{k!},\qquad f\in C[0,\infty),$$

the Baskakov operators

$$V_n(f;x)=\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\binom{n+k-1}{k}x^k(1+x)^{-n-k},\qquad f\in C[0,\infty),$$

and related ones.

We assume that L_n leaves linear functions invariant, i.e.,

$$L_n(t-x;x)\equiv 0,$$

and that

$$L_n((t-x)^2;x)=\alpha_n\varphi^2(x)\tag{2}$$

with some function φ, e.g., $\alpha_n=1/n$, $\varphi(x)=\sqrt{x(1-x)}$ for the Bernstein operators, $\alpha_n=1/n$, $\varphi(x)=\sqrt{x}$ for the Szász–Mirakjan operators, and $\alpha_n=1/n$, $\varphi(x)=\sqrt{x(1+x)}$ for the Baskakov operators (cf. [2, Chap. 9]).

We assume that the sequence $\{\alpha_n\}$ is decreasing and satisfies

$$\alpha_{Kn} \leqslant \tfrac{1}{2}\alpha_n, \qquad n = 1, 2, ... \tag{3}$$

with some K. Besides these we need some inequalities involving the derivatives of the L_n, namely

$$\|\varphi^2 L_n''(f)\| \leqslant C\, \|\varphi^2 f''\|, \tag{4}$$

$$\|\varphi^2 L_n''(f)\| \leqslant \frac{C}{\alpha_n}\, \|f\|, \tag{5}$$

$$\|\varphi^3 L_n'''(f)\| \leqslant \frac{C}{\alpha_n^{3/2}}\, \|f\|, \tag{6}$$

$$\|\varphi^3 L_n'''(f)\| \leqslant \frac{C}{\sqrt{\alpha_n}}\, \|\varphi^2 f''\| \tag{7}$$

with the appropriate smoothness assumptions on $L_n(f)$ and f. Each of these is well known (see [2, Chap. 9]) for the classical operators mentioned above with the possible exception of (7) which is a variation on the inequality (6) and can be obtained with the same methods (see in [2, Sect. 9.7]).

We need one more technical assumption on L_n, which is, however, a very natural one in view of the positivity of L_n and the moment condition (2). This is the following: let $a > 0$ be fixed and

$$E_m = E_{a,m} = \{x_0 \mid x_0 \pm a\sqrt{\alpha_m}\,\varphi(x_0) \in I\},$$

$$g_{M,m,x_0}(t) = \begin{cases} (t - x_0)^2 & \text{if } |t - x_0| \geqslant M\sqrt{\alpha_m}\,\varphi(x_0) \\ 0 & \text{otherwise.} \end{cases}$$

Then[1]

$$L_m(g_{M,m,x_0}; x_0)/(\alpha_m \varphi^2(x_0)) \to 0 \tag{8}$$

as $M \to \infty$ uniformly in m and $x_0 \in E_{a,m}$. This condition is again well known for the classical operators mentioned above. If we compare it with (2), we can see that it claims a certain localization for L_n.

The corresponding modulus of smoothness is defined as in (1) except that now the second supremum is taken for those values x for which every argument belongs to I. As usual, we need some restrictions on φ (see

[1] It is tacitly assumed that L_n is defined for the non-continuous function g_{M,m,x_0}.

[2, Sect. 1.2]); namely we assume that it is continuous and positive, and at the endpoints of I it behaves like a power of the distance from the endpoint: if a is a finite endpoint of I then

$$\varphi(x) \sim |x-a|^{\beta(a)} \qquad \text{as} \quad x \to a$$

with some $0 \leqslant \beta(a) < 1$, while if a is plus or minus infinity, then

$$\varphi(x) \sim |x|^{\beta(a)} \qquad \text{as} \quad x \to a$$

with some $0 \leqslant \beta(a) \leqslant 1$. These are again trivially satisfied in the cases discussed above.

With these assumptions we can now prove

THEOREM 1. *Suppose the positive operators $\{L_n\}$ satisfy the above conditions. Then there exist two constants K_1 and C_1 such that for all $f \in C(I)$ and n*

$$\|L_n(f)-f\| \leqslant C_1 \omega_\varphi(f; \sqrt{\alpha_n}), \tag{9}$$

and for $m \geqslant K_1 n$

$$\omega_\varphi(f; \sqrt{\alpha_n}) \leqslant C_1 \frac{\alpha_n}{\alpha_m} (\|L_n(f)-f\| + \|L_m(f)-f\|). \tag{10}$$

Proof. In the proof below, C denotes constants that depend exclusively on φ and the constants C above.

The proof of (9) is standard, but for completeness we include it. The assumptions on φ easily imply that for twice continuously differentiable g

$$|g(t)-g(x)-g'(x)(t-x)| \leqslant C \frac{(t-x)^2}{\varphi^2(x)} \|\varphi^2 g''\| \tag{11}$$

and hence

$$\left| g(t)-g(x)-g'(x)(t-x)-\frac{1}{2} g''(x)(t-x)^2 \right| \leqslant C \frac{(t-x)^2}{\varphi^2(x)} \|\varphi^2 g''\| \tag{12}$$

for all x, $t \in I$, which, combined with the positivity of L_n and (2), implies

$$\|L_n(g)-g\| \leqslant C\alpha_n \|\varphi^2 g''\|. \tag{13}$$

Thus,

$$\|L_n(f)-f\| \leqslant \inf_g (\|f-g\| + C\alpha_n \|\varphi^2 g''\|),$$

and it is well known (see [2, Theorem 2.1.1]) that the right-hand side is equivalent to $\omega_\varphi(f; \sqrt{\alpha_n})$.

Using the last remark the estimate

$$\alpha_n \|\varphi^2 L_n''(f)\| \leqslant C \frac{\alpha_n}{\alpha_m} (\|L_n(f)-f\| + \|L_m(f)-f\|) \tag{14}$$

proves (10).

Fix a $B \geqslant 1$ that is specified below. If

$$\|L_n(f)-f\| \leqslant \frac{\alpha_n}{B} \|\varphi^2 L_n''(f)\| \tag{15}$$

is not satisfied, then (14) is clearly true, so in what follows we may assume (15).

We get from (6), (7), and (15) (for any $B \geqslant 1$) that

$$\begin{aligned} \|\varphi^3 L_n'''(f)\| &\leqslant \|\varphi^3 (L_n \circ L_n(f) - L_n(f))'''\| + \|\varphi^3 (L_n \circ L_n(f))'''\| \\ &\leqslant \frac{C}{\alpha_n^{3/2}} \|L_n(f)-f\| + \frac{C}{\sqrt{\alpha_n}} \|\varphi^2 L_n''(f)\| \\ &\leqslant \frac{C}{\sqrt{\alpha_n}} \|\varphi^2 L_n''(f)\|. \end{aligned} \tag{16}$$

Let us choose a small $a > 0$ to be specified below and consider the set E_n of those $x \in I$ for which $x \pm a\sqrt{\alpha_n}\,\varphi(x) \in I$. For the sake of simplicity we assume that E_n is an interval, but the consideration below can be easily modified when this is not the case. Our next aim is to show that by appropriately choosing $a > 0$ (and $B > 1$) the supremum norm of $L_n(f)$ on E_n is comparable to that on the whole I. Let h_n be the twice continuously differentiable function that coincides with $L_n(f)$ on E_n and is a polynomial of degree at most two on the two intervals of $I \setminus E_n$. Equation (12) applied to the two endpoints of E_n immediately implies that

$$\|h_n - L_n(f)\| \leqslant C a^2 \alpha_n \|\varphi^2 L_n''(f)\|.$$

Hence we obtain from (4) and (5)

$$\begin{aligned} \|\varphi^2 L_n''(f)\| &\leqslant \|\varphi^2 L_n''(f-h_n)\| + \|\varphi^2 L_n''(h_n)\| \leqslant \frac{C}{\alpha_n} \|f-h_n\| + C\|\varphi^2 h_n''\| \\ &\leqslant \frac{C}{\alpha_n} (\|L_n(f)-h_n\| + \|L_n(f)-f\|) + C\|\varphi^2 h_n''\| \\ &\leqslant C a^2 \|\varphi^2 L_n''(f)\| + \frac{C}{B} \|\varphi^2 L_n''(f)\| + C\|\varphi^2 L_n''(f)\|_{E_n}, \end{aligned}$$

where we have also used (15) and that the assumptions on φ imply that the value of φ^2 on the two subintervals of $I\backslash E_n$ is at most a constant times its value at the two endpoints of E_n; hence we have the inequality

$$\|\varphi^2 h_n''\|_I \leqslant C\, \|\varphi^2 L_n''(f)\|_{E_n}$$

because the second derivative of h_n is constant (and equals the second derivative of $L_n(f)$ at the endpoints of E_n) on the two subintervals of $I\backslash E_n$. Thus, by choosing $a>0$ and $B\geqslant 1$ appropriately we can see that

$$\|\varphi^2 L_n''(f)\|_I \leqslant C_0\, \|\varphi^2 L_n''(f)\|_{E_n}. \tag{17}$$

Hence, there exists an $x_0 \in E_n$ such that, say,

$$\varphi^2(x_0)\, L_n''(f; x_0) \geqslant \frac{1}{C_0}\, \|\varphi^2 L_n''(f)\|.$$

Since on E_n we are "far" from the finite endpoint(s) of I, (16) easily implies that then

$$\varphi^2(x_0)\, L_n''(f; t) \geqslant \frac{1}{2C_0}\, \|\varphi^2 L_n''(f)\|$$

is also satisfied provided $|t-x_0| \leqslant \sqrt{\alpha_n}\, \varphi(x_0)/A$ with some $A>1$ independent of f and n. Hence,

$$L_n(f; t) - L_n(f; x_0) - L_n'(f; x_0)(t-x_0) \geqslant \frac{1}{4C_0}\, \|\varphi^2 L_n''(f)\|\, \frac{(t-x_0)^2}{\varphi^2(x_0)}$$

for $|t-x_0| \leqslant \sqrt{\alpha_n}\, \varphi(x_0)/A$. For other t we apply (11), which, together with the preceding inequality yields that for every t

$$L_n(f; t) - L_n(f; x_0) - L_n'(f; x_0)(t-x_0) \geqslant \|\varphi^2 L_n''(f)\|\, H_n(t), \tag{18}$$

where

$$H_n(t) = \begin{cases} (t-x_0)^2/4C_0 & \text{if } \; |t-x_0| \leqslant \sqrt{\alpha_n}\, \varphi(x_0)/A \\ -C_1(t-x_0)^2/\varphi^2(x_0) & \text{otherwise,} \end{cases}$$

and C_1 is a constant that does not depend on f or n and is chosen so large that the preceding inequality holds (see (11)).

Now all we have to do is to apply the operator L_m, $m > n$ to the last inequality. It follows from (3) and (8) that there is a $K_1 > 0$ such that if $m \geqslant K_1 n$, then

$$L_m(H_n; x_0) \geqslant \frac{\alpha_m}{8C_0} \|\varphi^2 L_n''(f)\|,$$

which, together with (18), yields

$$\|L_m f - f\| + 2\,\|L_n(f) - f\| \geqslant \|L_m(L_n(f)) - L_n(f)\| \geqslant \frac{\alpha_m}{8C_0} \|\varphi^2 L_n''(f)\|,$$

and this proves (14). ∎

It may happen that the operators L_n do not possess the necessary smoothness required by the inequalities (4)–(7), but some iterates of them do satisfy the analogues of these inequalities. This is the case for example with the averaging operators discussed in [1]. In such cases the above argument can be applied to the corresponding iterates as was done by Ditzian and Ivanov. I would add, however, that Ditzian and Ivanov discussed some non-positive operators, as well, for which case the geometric approach outlined in this paper cannot work.

References

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2. Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer Series for Computational Mathematics, Vol. 9, Springer-Verlag, New York, 1987.